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A New Class of Exact Traveling Wave Solutions to the Klein-Gordon Equation

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Abstract

This article examines the Klein-Gordon (KG) equation with cubic nonlinearity by an ansatz method. Some new exact traveling wave solutions to the KG equation have been extracted in the simplest form through this ansatz. In this ansatz a nonlinear ordinary differential equation has been used as an auxiliary equation whose solutions are identified. Physical significance as well as graphical representations of the extracted solutions has been discussed in a comprehensive fashion.

Keywords: Klein-Gordon (KG) equation, Solitary wave solutions, Periodic wave solutions.

1. Introduction

In the recent years, the exact solutions of nonlinear partial differential equations have been investigated by many researchers who are involved in nonlinear phenomena which exist in all fields including either the systematic works or engineering fields, such as, plasma physics, fluid mechanics, chemical physics, chemical kinematics, elastic media, optical fibers, solid state physics, biology, atmospheric and oceanic phenomena and so on. The research of traveling wave solutions of some nonlinear evolution equations (NLEEs) derived from such fields play an important role in the analysis of these phenomena. To obtain traveling wave solutions, many effective methods have been presented in the literature, such as, the $exp(-\varphi(\eta))$-expansion method [1,2], the $(G'/G, 1/G)$-expansion method [3], the (G'/G)-expansion method [4-10], the inverse scattering transform method [11], the Exp-function method [12,13], the Cole-Hopf transformation method [14], the Adomian decomposition method [15], the homotopy perturbation method [16], the Kudryashov method [17], the new approach of generalized (G'/G)-expansion method [18-20], the improved (G'/G)-expansion method [21], the tanh-function method [22], the tanh-coth method [23], the ansatz method [24], the novel (G'/G)-expansion method [25,26] and so on.

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The Klein-Gordon (KG) equation is an important NLEE that arise in relativistic quantum mechanics and quantum field theory, which is also much important for the high energy particle physics and is used to model many types of phenomena, including the propagation of dislocations in crystals and the behavior of elementary particles. The most common generalization of the linear theory above is to add a scalar potential $V(\varphi)$ to the equations of motion, where typically, V is a polynomial in φ of order 3 or more. Such a theory is sometimes said to be interacting, because the Euler-Lagrange equation is now nonlinear, implying a self-interaction. The action for the most general such theory is

$$S = \int d^{n-1} x dt \mathscr{L}, \text{ where } \mathscr{L} = \frac{1}{2}(\partial,\phi) - \frac{1}{2}\delta^{ij}(\partial,\phi)(\partial,\phi) - V(\phi).$$

The corresponding Euler-Lagrange equation of motion is $\partial_t^2\phi - \nabla^2\phi + V'(\phi) = 0$. If we take

$$V(\phi) = \frac{1}{2}m^2\phi^2 + \frac{\lambda^4}{4!}\phi^4 \quad ,$$

then we can easily derive the nonlinear KG equation form the corresponding Euler-Lagrange equation of motion . There is an amount of paper [27-33], where the various types of nonlinear KG equations are studied. Chowdhury and Biswas [32] studied the singular solitons and numerical analysis of the Phi-four equation $q_{tt} - k^2 q_{xx} = aq + bq^3$ that appears in relativistic quantum mechanics. The Phi-four equation is a special case of the Klein-Gordon equations

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that is studied with several forms of nonlinearity that includes quadratic nonlinearity, power law nonlinearity, as well as log law nonlinearity. Biswas et al. [33] also studied the solitons and conservation law of the KG equation with power law and log law nonlinearities. It is primarily the perturbation theory, numerical simulation, and integrability issues that have been addressed thus far in such models. If we set $k = 1$, $a = \alpha$, $b = -\beta$, then the Phi-four equation can be reduced to the KG equation with cubic nonlinearity $u_{tt} - u_{xx} + \alpha u + \beta u^3 = 0$ for one dimensional space time which is found in the literature [34,35].

The aim of this article is to explore a new class of exact travelling wave solutions to the KG equation by an ansatz method that appeared in recent time. In this method we consider a non-linear ordinary differential equation (ODE) as an auxiliary equation whose solutions are known [24]. The advantage of the proposed method over the existing method is that it provides new exact traveling wave solutions together with additional free parameters. The exact solutions have great values to unveil the inner structure of the physical phenomena. Apart from the physical significance, the close-form solutions of NLEEs help the numerical solvers to compare the correctness of their results and help them in the stability analysis. Algebraic manipulations of the proposed scheme are so simple that it does not need any software which is the main advantage of this method than the other existing methods.

The rest of the article is organized as follows: In Section 2, the description of the method is given. In Section 3, we apply this method to the NLEE pointed out above. The physical explanations and graphical representations of the obtained solutions are presented in Section 4. In Sections 5, we draw our conclusions.

2. Methodology

Let us consider a general nonlinear PDE in the form

$$F(u, u_t, u_x, u_{xx}, u_{tt}, u_{tx}, \ldots). \quad (1)$$

where $u = u(x,t)$ is an unknown function, F is a polynomial in $u(x,t)$ and its derivatives in which highest order derivatives and nonlinear terms are involved and the subscripts stand for the partial derivatives. The main steps of this method are as follows:

Step 1: Combine the real variables x and t by a complex variable ζ as

$$u(x,t)=U(\zeta), \text{ where } \zeta=x+ct \quad (2)$$

where c is the speed of the traveling wave. The traveling wave transformation (2) converts Eq. (1) into an ordinary differential equation (ODE) for $U(\zeta)$:

$$R(U, Ut, Utt, Uttt \ldots \ldots) = 0 \quad (3)$$

where R is a polynomial of U and its derivatives and the superscripts indicate the ordinary derivatives with respect to ζ.

Step 2: Suppose the traveling wave solution of Eq. (3) can be expressed as follows:

$$U(\zeta)=\Sigma_{i=0}^{n} a_i V^i \quad (4)$$

where $(0 \le i \le n)$ are constants to be determined, such that $a_n \neq 0$ and $V = V(\zeta)$ satisfies the following ordinary differential equation:

$$\left(\frac{dV}{d\zeta}\right)^2 = V^2(\lambda + \mu V^2) \quad (5)$$

where, λ and μ are nonzero constants.

Eq. (5) gives the following solutions:

Case-I: When $\lambda < 0$

$$\left.\begin{aligned} V &= \pm\sqrt{-\frac{\lambda}{\mu}}\sec[\sqrt{-\lambda}(\zeta+\zeta_0)] \\ V &= \pm\sqrt{-\frac{\lambda}{\mu}}\csc[\sqrt{-\lambda}(\zeta+\zeta_0)] \end{aligned}\right\} \quad (6)$$

where, ς_0 is the integrating constant.

Case-II: When $\lambda < 0$

$$\left.\begin{aligned} V &= \pm\sqrt{-\frac{\lambda}{\mu}}\operatorname{sech}[\sqrt{\lambda}(\zeta+\zeta_0)] \\ V &= \pm i\sqrt{-\frac{\lambda}{\mu}}\operatorname{csch}[\sqrt{\lambda}(\zeta+\zeta_0)] \end{aligned}\right\} \quad (7)$$

Step 3: Substitute Eq. (4) into Eq. (3) with the help of (5) and then we account the function $V = V(\zeta)$. As a result of this substitution, we get a polynomial equation of $V = V(\zeta)$. Then we will equate all the coefficients of the like powers of V to zero. This procedure yields a system of algebraic equations. After solving the systems for $a_0, a_1, \ldots, c, \lambda, \mu$ and substituting these values into Eq. (4) along with general solutions of Eq. (5) completes the determination of the solution of Eq. (1).

3. New Exact Solutions to the Klein-Gordon Equation

Let us consider the KG equation of the following form

$$u_{tt} - u_{xx} + \alpha u + \beta u^3 = 0 \quad (8)$$

where, $u(x.\ t)$ represents the particle wave profile at any varied instances and α, β are nonzero real constants.

If we combine x and t by a compound variable ς i.e. $u(x.\ t)=U(\varsigma)$, $\varsigma=x+ct$, and differentiating $u(x.\ t)$ partially with respect to x and t two times, that is, $u_{xx}=U''(\varsigma), u_{tt}=c^2U''(\varsigma)$, then the equation (8) reduces to the following nonlinear ODE:

$$(c^2 - 1)U'' + \alpha U + \beta U^3 = 0 \qquad (9)$$

Assuming the solutions of equation (9) as equation (4) and by balancing higher order derivative term with non-linear term that appeared in eq.(9), we obtain $n = 1$. Thus eq. (4) becomes

$$U(\varsigma) = a_o + a_1 V \qquad (10)$$

Substituting (10) in (9) by the help of (5), we get a polynomial equation of V and then equating the corresponding coefficients of like powers of V, we obtain the following algebraic system

$$\left.\begin{aligned} &2a_1\,\mu(c^2-1) + \beta a_1^3 = 0 \\ &3a_0\,\beta a_1^2 = 0 \\ &a_1\,\lambda(c^2-1) + a a_1 + 3a_1\,\beta a_0^2 = 0 \\ &a a_0 + \beta a_0^3 = 0 \end{aligned}\right\} \qquad (11)$$

Solving the algebraic system for a_o, a_1 and c, we obtain

$$\left.\begin{aligned} a_0 &= 0 \\ a_1 &= \pm\sqrt{\frac{2\mu\alpha}{\lambda\beta}} \\ c &= \pm\sqrt{1-\frac{\alpha}{\lambda}} \end{aligned}\right\} \qquad (12)$$

By combining equations (2), (6), (7), (10) and (12), the KG equation has the following explicit solutions as follows:

Set-1: When $\lambda < 0$

$$u_1(x,t) = \sqrt{-\frac{2\alpha}{\beta}}\sec[\sqrt{-\lambda}(x \pm \sqrt{1-\frac{\alpha}{\lambda}}t + \zeta_0)] \qquad (13)$$

$$u_2(x,t) = \sqrt{-\frac{2\alpha}{\beta}}\csc[\sqrt{-\lambda}(x \pm \sqrt{1-\frac{\alpha}{\lambda}}t + \zeta_0)] \qquad (14)$$

Set-2: When $\lambda < 0$

$$u_3(x,t) = \sqrt{-\frac{2\alpha}{\beta}}\operatorname{sech}[\sqrt{\lambda}(x \pm \sqrt{1-\frac{\alpha}{\lambda}}t + \zeta_0)] \qquad (15)$$

$$u_4(x,t) = i\sqrt{-\frac{2\alpha}{\beta}}\operatorname{csch}[\sqrt{\lambda}(x \pm \sqrt{1-\frac{\alpha}{\lambda}}t + \zeta_0)] \qquad (16)$$

These obtained solutions are very helpful to analyze the particle wave propagation in relativistic quantum mechanics and quantum field theory, which is also much important for the high energy particle physics. These solutions are also useful to describe the propagation of dislocations in crystals and the behavior of elementary particles.

4. Physical Explanations and Graphical Representations

In this section we will discuss the physical explanations and graphical representation of the above determined four families of the solutions.

The introduction of dispersion without introducing nonlinearity destroys the solitary wave as different Fourier harmonics start propagating at different group velocities. On the other hand, introducing nonlinearity without dispersion also prevents the formation of solitary waves, because the pulse energy is frequently pumped into higher frequency modes. Similarly to dispersion, dissipation can also give rise to solitary waves when combined with nonlinearity. A solitary wave is a wave which propagates without any temporal evolution in shape or size when viewed in the reference frame moving with the group velocity of the wave. The envelope of the wave has one global peak and decays far away from the peak. Solitary waves arise in many literatures, including the elevation of the surface of water, the intensity of light in optical fibers, the particle wave propagation in field theory, the elevation of surface in shallow water wave etc. However, if both dispersion and nonlinearity are present, solitary waves can be sustained. Similarly to dispersion, dissipation can also give rise to solitary waves when combined with nonlinearity. The KG equation incorporates two competing effects: (i) the nonlinear term represented by u^3 that describe the translations of wave and (ii) the linear dispersion term represented by u_{xx} that describes the spreads it out. If both dispersion and nonlinearity are present, solitary waves can be persistent. Hence it is more interesting to point out that the delicate balance between the nonlinearity effect of u^3 and the dissipative effect of u_{xx} give rise to solitons solitary waves, that after a fully interaction with others the solitons come back retaining their identities with the same speed and shape. A soliton is also a nonlinear solitary wave with the additional property that the wave retains its permanent structure, even after interacting with another soliton. For example, two solitons propagating in opposite directions effectively pass through each other without breaking. There are various types of solitary and periodic wave solutions that appeared from the analytical solutions to the NLEE by choosing appropriate values of the physical parameters. In this article, the solitary wave solutions originated from the explicit solutions to the KG equation for some special values of additional free parameters are given as follows:

Solution (13) represents the solitary wave solution of bright and dark soliton type to the KG equation. Fig. 1 below shows the bright and dark soliton type exact solitary wave solution of Eq. (13) with some fixed parametric values $\lambda = -0.1$, $\alpha = 0.5$, $\beta = -0.1$, $\varsigma_0 = 0.5$ and $-3 \le x,\ t \le 3$.

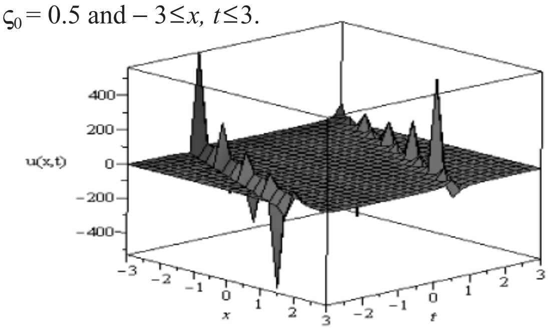

Fig. 1 Bright and dark soliton of exact solution (13) with $-3 \le x,\ t \le 3$.

Solution (14) also represents the solitary wave solution of bright and dark soliton type to the KG equation. Fig. 2 below shows the bright and dark soliton type exact solution of Eq. (14) with some fixed parametric values $\lambda = -0.5$, $\alpha = 0.01$, $\beta = -0.1$, $\varsigma_0 = 0$ and $-3 \le x,\ t \le 3$.

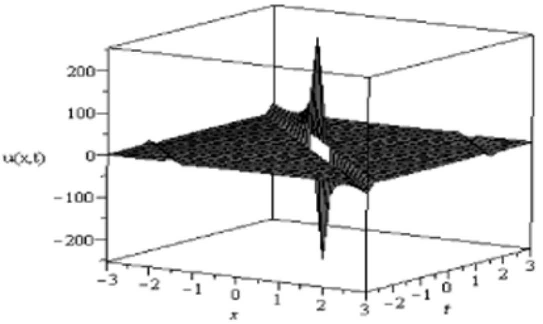

Fig. 2 Bright and dark soliton of exact solution (14) with $-3 \le x,\ t \le 3$.

Solution (15) represents the exact bell type soliton solution to the Klein-Gordon (KG) equation. Solitons are very special kinds of solitary waves which described many physical phenomena in soliton physics. The soliton solution is a specially localized solution, hence $u'(\zeta), u''(\zeta), u'''(\zeta) \to 0$ as $\zeta \to \pm\infty$, $= x+ct$. Solitons have a remarkable property- it keeps its identity upon interacting with other solitons. Soliton solutions also give rise to particle-like structures, such as magnetic monopoles etc. So, soliton are everywhere in the nature. Fig. 3 below show the exact solitary wave solution of bell type of Eq. (15) with some fixed parametric values $\lambda = 0.5$, $\alpha = -0.5$, $\beta = 0.5$, $\varsigma_0 = 0.5$ and $-3 \le x,\ t \le 3$.

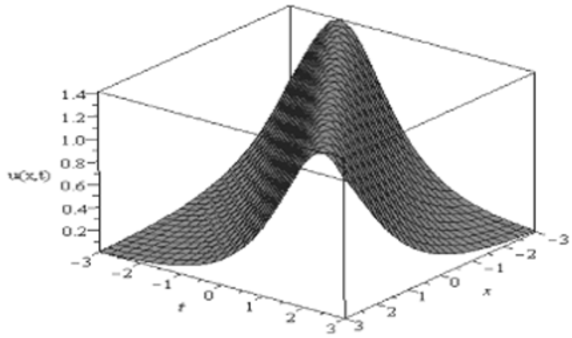

Fig. 3: Bell type solitary wave solution obtained from (15) with $-3 \le x,\ t \le 3$.

Solution (16) represents the solitary wave solution of double soliton type to the KG equation. Fig. 4 below shows the double soliton type exact solitary wave solution of Eq. (16) with some fixed parametric values $\lambda = 0.1$, $\alpha = 0.5$, $\beta = -0.5$, $\varsigma_0 = 0.5$ and $-3 \le x,\ t \le 3$.

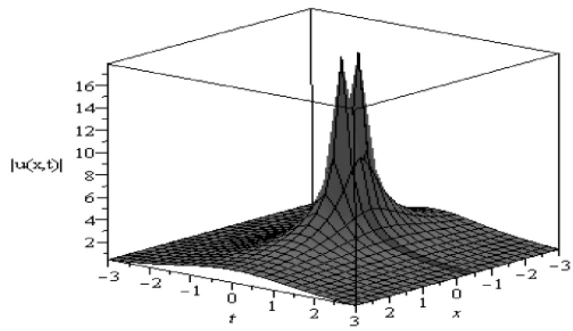

Fig. 4: Double soliton solution obtained from (16) with $-3 \le x, t \le 3$.

5. Conclusions

The ansatz method is successfully applied to establish the valuable explicit form traveling wave solutions to the famous KG equation. The performance of this method is reliable, convincing and can be used to other NLEEs in finding exact solutions. The method gives more general solutions which contain further arbitrary constants and the arbitrary constants imply that these solutions have rich local structures. The results revealed remarkable relations of solitary pattern solutions or solitons. Although the method has a lot of merit it has a few drawbacks, such as, sometimes the method gives solutions in disguised versions of known solutions that may be found by other methods. The obtained solutions can also be utilized to further analyze by physicists on varied instance.

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